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Monodromy properties of energy momentum tensor on general algebraic curves

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Abstract

A new approach to analyze the properties of the energy momentum tensor $T(z)$ of conformal field theories on generic Riemann surfaces (RS) is proposed. $T(z)$ is decomposed into N components with different monodromy properties, where N is the number of branches in the realization of RS as branch covering over the complex sphere. This decomposition gives rise to new infinite dimensional Lie algebra which can be viewed as a generalization of Virasoro algebra containing information about the global properties of the underlying RS. In the simplest case of hyperelliptic curves the structure of the algebra is calculated in two ways and its central extension is explicitly given. The algebra possess an interesting symmetry with a clear interpretation in the framework of the radial quantization of CFTs with multivalued fields on the complex sphere.

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1. Introduction

The problem of constructing conformal field theories (CFTs) on higher genus Riemann surfaces (RS) arose in the middle of the previous decade in the context of perturbative string theories, where higher loop computations require the evaluation of CFTs on higher genus RS

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[1]. CFTs have been studied in connection with phase transitions phenomena at criticality, where the physics becomes scale invariant. The structure of CFTs in $D = 2$ dimensions turns out to be particularly rich due to the infinite dimensional group of local conformal transformations generated by the Virasoro algebra [2]. The presence of this algebra allows the complete solution of certain conformal models [3]. It became very soon clear that the conformal symmetry plays a crucial role in understanding the structure of string theory. The possible consistent vacua for string theory can be recognized as CFTs [4].

In investigating CFTs it has been fruitful to study these theories on the torus. In particular, the idea of modular invariance has found many applications. Modular invariance involves left and right moving sectors simultaneously and provides severe restrictions on the class of "acceptable" CFTs [5]. The properties of CFTs on more general RS are perhaps not so relevant to physics, at least from the present point of view. Yet it is interesting to consider "new examples" of CFTs. One can talk about "new examples" in the following sense. Most of the models discussed so far are free field theories. They are however nontrivial due to the interaction with two-dimensional gravity and because of the topology of the manifolds on which the theories have been defined [6–8]. They can also be viewed as "new" since they can be treated as CFTs on complex sphere with multivalued conformal fields.

One of the most interesting approaches to analyze CFTs on higher genus RS is to define an equivalent multivalued CFT on the complex sphere. To do this, we notice that an RS can be represented as a branch covering of the sphere, i.e. it can be visualized as N copies of the complex sphere "glued" at some branch points [9]. The information about the geometry of the RS is encoded in its monodromy properties. In the language of the multivalued CFTs mentioned above, one can construct "twist operators" located at the projections of the branch points on the sphere, which "simulate" the effects of the geometry. In this way the CFT on the multisheeted RS is claimed to be equivalent to the theory on the complex sphere supplemented by extra twist field insertions. More precisely, since a CFT is usually defined by a set of correlation functions with required analytic properties, equivalence means here that the correlation functions on the sphere with twist fields are formally the same as the correlation functions on the RS.

A first paper containing similar ideas in a slightly different context of CFTs on orbifolds is that of Dixon et al. [10]. Later, several more detail papers followed [11–13]. In the case of the b–c systems on Z_N symmetric curves, a very clear picture was obtained in the language of bosonization: the twist operators have been explicitly constructed. The treatment of bosonic fields with quadratic action turns out to be more complicated, but still the practical purpose of calculating the combinations of chiral determinants entering in multiloop string amplitudes has been achieved [14]. The above program was stopped by the difficulty of dealing with generic curves outside the class of the Z_N symmetric ones. On the other hand, already at genus 3, the hyperelliptic curves represent a subvariety of codimension 1, i.e. a subset of measure zero in the moduli space. Also the interesting approaches to treat CFTs on general RS proposed in [15,16] have been abandoned. At the present, one can only speculate if it is possible to encode all the information about the perturbative expansion of string theory in the limit $N \to \infty$ of some CFT on the sphere (see formula (12.14) in [16]). In recent years some progress in dealing with CFT on general RS can nevertheless be reported. In particular, an operator formalism for CFTs on general RS has been constructed in [17,18] and minimal models of CFT has been defined on Z_N symmetric curves [19] (before only the hyperelliptic case was studied in this context [20]).

In this paper a modified approach to the problem of constructing CFTs on RS is put forward so that the global geometrical aspects can be taken into account. One obtains a new algebraic structure (infinite dimensional Lie algebra) which should be viewed as an analog of the Virasoro algebra. It is not the first time that such structure is proposed [21]. The algebra proposed in this paper is very different but there seems to be a common spirit which underlies both constructions. Our construction can be viewed as a kind of multipoint generalization of Krichever–Novikov basis: there are (provided they are not branch points) exactly N points (one on each sheet) with coordinate $z = 0$ and $z = \infty$. The elements of the basis in which the solutions of the classical equations of motion are expanded, are allowed to have poles only at these points [22].

Technically, the difficulties to deal with CFT on generic RS are related with their monodromy group. In the Z_N symmetric case all the monodromy matrices are "the same" due to the peculiar symmetry of the RS and can be simultaneously diagonalized. For general RS this is impossible. Take for example a generic curve of genus $g = 3$. This can be viewed as a three-sheeted covering of the complex sphere with ten branch points. Two sheets (for instance the second and third) are glued together at two branch points while the first and second sheets are glued at the remaining branch points. The monodromy matrices associated to the first two branch points are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and the remaining eight are

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly, they do not commute and so they cannot be simultaneously diagonalized. However, something else can be done: any analytic tensor field of spin λ (also called here meromorphic λ-differential) with given poles and zeros can be decomposed into a sum of independent components modulo single-valued functions with different monodromy properties. These components, whose number is equal to the number of sheets of the RS in question, can be explicitly constructed. Of course, there are many possible monodromy decompositions of this kind. From the point of view of the construction presented in this paper, all of them seems to be equivalent, but to make computations easier it is important to find the simplest possible decomposition. This point will be discussed in Section 5 in more details.

Assume that the RS is given by means of algebraic equation (polynomial) in two complex variables z and y:

$$F(z, y) = y^N P_N(z) + y^{N-1} P_{N-1}(z) + \cdots + P_0(z) = 0, \tag{1.1}$$

where $P_j(z)$ are polynomials in z. y is a single-valued function on the RS but one can also consider $y(z)$ as multivalued function on the complex sphere. The monodromy properties of $y(z)$ define the RS. We shall adopt the notation $y^{(j)}(z)$, where $j = 1, 2, \ldots, N$ refers to the value of y on the j branch of the RS. z plays a double role in the construction. It can be viewed either as single-valued function on the RS or as a coordinate (with trivial monodromy properties) on the complex sphere on which the N sheets composing the RS are projected. Transporting the function $y(z)$ along a small closed path around a given branch point located at $z = a$, its branches are exchanged according to the local monodromy properties of the RS in a. Denoting with $m_a[f]$ the operator which transport a multivalued function $f(z)$ around a, it is easy to see that

$$m_a(y^{(j)}(z)) = \sum_{k=1}^{N} M_{(a)}^{jk} y^{(k)}(z), \tag{1.2}$$

where the $M_{(a)}^{jk}$ are N-dimensional permutation matrices. All the meromorphic tensor fields can be constructed in terms of powers of $\mathrm{d}z$ and rational functions of z and y [9]. It is clear that the nontrivial monodromy is carried only by y. The simplest monodromy decomposition of a tensor field ω being a meromorphic λ-differential is given by

$$\omega(z) = \sum_{k=0}^{N-1} g_k(z) y^k(z)\, \mathrm{d}z^\lambda, \tag{1.3}$$

where $g_k(z)$ are rational functions of z. An elementary proof of that fact can be found in [18].

The monodromy decomposition is a basic idea behind the construction of the operator formalism on RS in [17,18]. Usually by CFTs on RS one understands a set of correlation functions satisfying necessary analytic properties. These "physical" requirements are typically strong enough to define correlators up to an overall normalization [23].

In all approaches to establish an operator formalism for CFTs on RS, one tries to construct a vacuum state and to act on it with annihilation and creation operators as in ordinary quantum field theories in flat spaces, see e.g. [24]. In the approach of [18], the b–c fields are represented in a suitable way in monodromy decomposition form. Two different expansions are chosen for b and c. In this way the simplest possible anticommutation relations between the elementary excitations of both fields can be postulated.

On arbitrary genus RS defined by means of (1.1) such decompositions are given by

$$b(z) = \sum_{k=0}^{N-1} \sum_{i=-\infty}^{+\infty} b_{k,i} z^{-i-\lambda} f_k(z), \tag{1.4}$$

$$c(z) = \sum_{k=0}^{N-1} \sum_{i=-\infty}^{+\infty} c_{k,i} z^{-i+\lambda-1} \phi_k(z). \tag{1.5}$$

The nontrivial monodromy is carried by the (multivalued) functions

$$f_k(z) = \frac{y^{N-1-k}(z)\, \mathrm{d}z^\lambda}{(F_y(z, y(z)))^\lambda} \tag{1.6}$$

and

$$\phi_k(z) = \frac{dw^{1-\lambda}}{(F_y(z, y(z)))^{1-\lambda}}(y^k(z) + y^{k-1}(z)P_{N-1}(z) + \cdots + P_{N-k}(z)), \tag{1.7}$$

where $F_y \equiv \partial F/\partial y$. The single-valued functions multiplying them are powers of z as in the usual Laurent expansion. Except for the monodromy contributions we follow as close as possible the way in which the CFTs on the complex sphere are treated. To go to the quantum theory we postulate the following anticommutation relations:

$$\{b_{s,j}, c_{r,k}\} = \delta_{s,r}\delta_{j+k,0}. \tag{1.8}$$

After explaining in [18] all the details concerning the construction of vacuum state, splitting of oscillators into annihilation, creation and zero mode part, it is natural to continue the same analysis in the case of energy momentum tensor. This is an important object, because it generates the conformal transformations and its vacuum expectation value contains information about the dependence of the partition function of the theory on the moduli.

The monodromy components of the energy momentum tensor for b–c system can be expressed in terms of the elementary excitations $b_{k,i}$ and $c_{k,i}$ of the fields b and c. Once they are known, the commutators of the resulting operators can then be calculated. One obtains an infinite dimensional Lie algebra. More precisely, at this stage of the computations one disregards central extension terms related to normal ordering ambiguities. It will be shown that there is a simpler way to include these terms. It turns out that the algebra can be understood in terms of an operator product expansion (OPE) for the monodromy components of $T(z)$. This is one of two most important results of this paper. In deriving it, the usual techniques (radial quantization) of the operator formalism for CFT on the complex sphere are applied. The language of OPE makes it possible to calculate also a central extension of the algebra.

In order to illustrate in details the above ideas the simpler hyperelliptic case is studied in Section 2. The monodromy decomposition of $T(z)$ gives rise to generators $L_{1,n}$ and $L_{2,m}$. Their Lie algebra brackets are explicitly computed. In the standard operator formalism on the sphere the conjugation of conformal fields is understood in the framework of the "radial quantization", where $z = 0$ corresponds to asymptotic "in" states and $z = \infty$ to asymptotic "out" states. Time inversion is expressed as $z \to 1/\bar{z}$. In the case of the elementary excitations for the fields b and c on RS our natural construction of the representation space leads itself to a definition of conjugation [25]. In Section 3 it is shown that this definition is equal to the one suggested by the radial quantization prescription. One finds that also the expressions of the operators $L_{1,j}$ and $L_{2,j}$ in terms of the multivalued modes in which the fields b and c have been expanded are subjected to a conjugation consistent with radial quantization considerations. This is the second important result of the paper. A surprising feature of the conjugation is that it involves parameters of the algebraic equation defining RS. The origin of this dependence is clearly explained.

One can try to reconstruct the generators l_n of the Virasoro algebra as infinite linear combination of $L_{1,n}$ and $L_{2,n}$. In the hyperelliptic case such representation is given but this procedure should be treated with caution. It is unclear if it can be generalized to the case of arbitrary algebraic curves. Also, the reconstructed l_n's do not have the property $l_n^\dagger = l_{-n}$.

The motivation of our investigations is the possibilty of finding analogous structures associated also to arbitrary RS. It is always possible to find the monodromy decomposition of $T(z)$ and postulate simple OPE for its monodromy components. There is some freedom in the form of the OPE so that there are free numerical parameters in the infinite dimensional Lie algebra. Performing in $F(z, y)$ the transformation $z \to 1/\bar{z}$ it is possible to introduce a natural notion of conjugation. The hope is that, thanks to the study of the detailed structure of this algebra one can get information about the physical content of the theory. In the case of the Virasoro algebra, the theory of its representations was a basic tool to investigate structure of CFT [3]. It is important that one can always perform explicit computations following those presented in Section 2 and compare them with the structure produced by OPE.

The new algebraic structure should be studied in more detail. One should address questions about the definition of primary fields, the general properties of representations, null states, unitary representations, etc. All these issues will be discussed in future papers.

2. Algebra in the hyperelliptic case

Let us consider the b–c systems with action

$$S_{bc} = \frac{1}{\pi} \int\limits_{RS} \mathrm{d}^2 z (b\bar{\partial} c + \text{c.c.}), \tag{2.1}$$

where z and $\bar{z}$ are complex coordinates on the RS, $\bar{\partial} \equiv \partial/\partial\bar{z}$. b and c are fermionic fields carrying conformal weights λ and $1-\lambda$ with classical equations of motion

$$\bar{\partial} b = \bar{\partial} c = 0. \tag{2.2}$$

The energy momentum tensor is

$$T(z) = -\lambda b(\partial c) + (1-\lambda)(\partial b)c. \tag{2.3}$$

Here we investigate the special case of hyperelliptic curves defined by the equation (∞ is not a ramification point):

$$y^2 = \prod_{j=1}^{2g+2} (z - a_j). \tag{2.4}$$

In accordance with [17,18] the following monodromy decomposition of both fields are postulated:

$$b(z) = \sum_{k=0}^{1} \sum_{j=-\infty}^{+\infty} b_{k,j} \frac{z^{-\lambda-j}}{y^{\lambda-k}} \mathrm{d}z^{\lambda}, \tag{2.5}$$

$$c(z) = \sum_{k=0}^{1} \sum_{j=-\infty}^{+\infty} c_{k,j} \frac{z^{\lambda-j-1}}{y^{k-\lambda}} \mathrm{d}z^{1-\lambda}. \tag{2.6}$$

One recognizes that they represent particular cases of (1.4) and (1.5). The elementary excitations satisfy the relation (see (1.8))

$$\{b_{s,j}, c_{r,k}\} = \delta_{s,r}\delta_{j+k,0}. \tag{2.7}$$

The following notation will be useful:

$$y^2 = \sum_{j=0}^{2g+2} A(j)z^j. \tag{2.8}$$

It will be understood that

$$A(j) = 0 \quad \text{for } j > 2g+2. \tag{2.9}$$

Let us introduce the following monodromy decomposition of the energy momentum tensor:

$$T(z) = \sum_{k=1}^{2} T_k(z) = \sum_{k=1}^{2} \sum_{j=-\infty}^{+\infty} L_{k,j} \frac{z^{-j-2}}{y^k}\, \mathrm{d}z^2. \tag{2.10}$$

The reason to take $k = 1, 2$ rather then $k = 0, 1$ is merely technical. With the adopted choice it is simpler to express $L_{k,j}$ in terms of $b_{r,n}$ and $c_{s,m}$.

In the hyperelliptic case in which the monodromy group can be diagonalized one should distinguish between the above "monodromy decomposition" and the treatment of $T(z)$ in [13], where the values of the energy momentum tensor on two branches of the curve: $T^{(k)}$, $k = 1, 2$ are arranged into linear combinations which are diagonal with respect to the monodromy group

$$T^{\pm}(z) = \frac{1}{\sqrt{2}}(T^1(z) \pm T^2(z)). \tag{2.11}$$

Thus in the notation explained in Section 1, the elements of the monodromy matrices are

$$M^{++} = -M^{--} = 1, \qquad M^{+-} = M^{-+} = 0. \tag{2.12}$$

Comparing Eqs. (2.10) and (2.11) it seems possible to identify T^+ with T_2 and T^- with T_1. This is because T_2 has trivial monodromy properties (y^2 can be expressed as polynomial in z). That correspondence is however misleading. By adding the components of $T(z)$ in both approaches one obtains

$$T_1 + T_2 = T, \qquad T^+ + T^- = \sqrt{2}T^1. \tag{2.13}$$

A correct way of thinking about $T^{\pm}$ is that of fields on $\mathbb{C}P(1)$ while T_k should rather be treated as fields on RS.

The advantage of the representation (2.10) is that it can be generalized to the case of general algebraic curve when the monodromy group cannot be globally diagonalized.

The energy momentum tensor (2.3) can be expressed in terms of the b and c field modes introduced in (2.5) and (2.6). By looking at the monodromy decomposition of $T(z)$ it is possible to express $L_{k,m}$ in terms of $c_{s,n}$ and $b_{r,m}$:

$$L_{2,p} = \sum_{k=0}^{1} \sum_{m=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} q_{p,m,j}^{(2-k)} : b_{k,j} c_{k,m-j}, \tag{2.14}$$

$$L_{1,p} = \sum_{k=0}^{1} \sum_{m=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} s_{p,m,j}^{(2-k)} : b_{k,j} c_{1-k,m-j}, \tag{2.15}$$

where

$$q_{p,m,j}^{(2)} = A(m-p)\left(\frac{\lambda}{2}(m+p) - j\right), \tag{2.16}$$

$$q_{p,m,j}^{(1)} = A(m-p)\left(\frac{\lambda}{2}(p+m) + \frac{1}{2}(m-p) - j\right), \tag{2.17}$$

$$s_{p,m,j}^{(2)} = \delta_{p,m}(\lambda p - j), \tag{2.18}$$

$$s_{p,m,j}^{(1)} = A(m-p)\left(\lambda p - j + \frac{1}{2}(m-p)\right). \tag{2.19}$$

The definition of normal ordering :: will be given in Section 4. It follows from (2.16)–(2.19) that sums over m in (2.14) and (2.15) are finite (see (2.9)). It is possible to calculate the algebra commutators. At this stage the full structure will not be obtained, namely central extension terms will be omitted. There is much easier way to calculate them.

Commutators of the algebra generators are:

$$[L_{2,p}, L_{2,r}] = (p-r) \sum_{s=p+r}^{p+r+2g+2} A(s-p-r) L_{2,s} + \text{c.e. terms}, \tag{2.20}$$

$$[L_{1,n}, L_{1,m}] = (n-m) L_{2,n+m} + \text{c.e. terms}, \tag{2.21}$$

$$[L_{2,p}, L_{1,n}] = \sum_{s=n+p}^{n+p+2g+2} \frac{1}{2}(s+p-3n) A(s-n-p) L_{1,s} \tag{2.22}$$

(in $L_{1,n}$ there is no normal ordering ambiguity). It will be shown in Section 4 that the normal ordering involves only $L_{2,n}$ with $n \leq 0$. It follows that central extension terms are present only for $p + r \leq 0$ in (2.20) and $n + m \leq 0$ in (2.21).

3. OPE for the monodromy components of $T(z)$

In the approaches of [11,13,16,19,20] the OPE for diagonal (with respect to monodromy) components of $T(z)$ are postulated. They have the property to keep the monodromy, i.e. the monodromy properties (which are described by single numbers, "charges") of both sides of OPE are the same. This can be adopted as a principle according to which more general situations are handled. For this purpose the identification of components of $T(z)$ in the way explained in (2.10) is very useful.

The requirement of unchanged monodromy plus self-consistency of the whole structure leads to the following OPEs (Γ, Γ', D, E, F, G, H are constants to be fixed):

$$T_2(z)T_2(w) \sim \frac{\Gamma'}{(z-w)^4} + \frac{2DT_2(w)}{(z-w)^2} + \frac{D\partial T_2(w)}{z-w}, \tag{3.1}$$

$$T_1(z)T_1(w) \sim \frac{\Gamma}{(z-w)^4} + \frac{2HT_2(w)}{(z-w)^2} + \frac{H\partial T_2(w)}{z-w}, \tag{3.2}$$

$$T_2(z)T_1(w) \sim \frac{2ET_1(w)}{(z-w)^2} + \frac{F\partial T_1(w)}{z-w}, \tag{3.3}$$

$$T_1(z)T_2(w) \sim \frac{2ET_1(w)}{(z-w)^2} + \frac{G\partial T_1(w)}{z-w}. \tag{3.4}$$

In the above OPEs the symmetry properties under the transformation $z \leftrightarrow w$ have been used. For example, in (3.1) one could have started with

$$T_2(z)T_2(w) \sim \frac{\Gamma'}{(z-w)^4} + \frac{2DT_2(w)}{(z-w)^2} + \frac{\tilde{D}\partial T_2(w)}{z-w} \tag{3.5}$$

but a simple analysis leads to condition $D = \tilde{D}$.

The standard OPE for $T(z) = T_1(z) + T_2(z)$ implies

$$D + H = 2E = F + G = 1 \tag{3.6}$$

and

$$\Gamma + \Gamma' = \frac{c}{2} = -(6\lambda^2 - 6\lambda + 1), \tag{3.7}$$

where c is the conformal anomaly of CFT. The operators $\tilde{L}_{k,n}$ can be expressed in terms of $T(z)$ via contour integrals. We find it useful to distinguish here $L_{k,n}$ expressed in terms of elementary excitations $b_{k,n}$ and $c_{k,n}$ from $\tilde{L}_{k,n}$ as, a priori, they need not satisfy the same algebra:

$$\tilde{L}_{k,j} = \frac{1}{\pi \mathrm{i}} \oint T_k y^k z^{j+1}\,\mathrm{d}z. \tag{3.8}$$

It is crucial to notice that the integral (3.8) is well defined even if T_k takes different values on different branches of the RS. The combination $T_k y^k$ (no summation in the index k!) is well defined on $\mathbb{C}P(1)$ as it has trivial monodromy properties. The origin of unusual normalization can be traced back in (2.13).

Using standard techniques of CFT (radial quantization etc.) one obtains

$$[\tilde{L}_{2,p}, \tilde{L}_{2,r}] = 2D(p-r) \sum_{s=p+r}^{p+r+2g+2} A(s-p-r)\tilde{L}_{2,s}$$

$$+\frac{2\Gamma'}{3}\sum_{s=\max(r,-p-2g-2)}^{\min(-p,2g+2+r)} A(-r+s)A(-p-s)$$

$$((s+p-r)^3-(s+p-r)). \tag{3.9}$$

A central extension term is present if $-(4g+4)\leq p+r\leq 0$.

$$[\tilde{L}_{1,n},\tilde{L}_{1,m}]=2H(n-m)\tilde{L}_{2,n+m}+\frac{2\Gamma}{3}\left(\frac{1}{2}A(-n-m)(n^3-m^3+m-n)\right.$$

$$-\frac{3}{8}\left(\sum_{j=1}^{\min(2g+2,-n-m)}\sum_{k=1}^{\min(2g+2,-n-m-j)}(n-m)jkA(j)A(k)\right.$$

$$\left.\left.\times\ B(-n-m-j-k)\right)\right). \tag{3.10}$$

Here again a central extension term is present for $-(4g+4)\leq n+m\leq 0$. Finally we have

$$[\tilde{L}_{2,p},\tilde{L}_{1,n}]=\sum_{s=n+p}^{n+p+2g+2}(s(4E-3F)+pF+n(F-4E))A(s-n-p)\tilde{L}_{1,s}. \tag{3.11}$$

In the above formulae the $B(s)$'s, $(s\geq 0)$, are defined by means of the relations

$$\sum_{n=\max(0,m-2g-2)}^{m\geq 0} B(n)A(m-n)=\delta_{m,0} \tag{3.12}$$

for $s'<0$, $B(s')=0$. In addition to central extension terms the generators $\tilde{L}_{k,j}$ satisfy the same algebra as $L_{k,j}$ provided that $D=H=F=E=\frac{1}{2}$. It is interesting to notice that the two numerical constants Γ' and Γ are not independent. The Jacobi identities for the whole structure imply

$$\Gamma'=\Gamma. \tag{3.13}$$

In the explicit verification of Jacobi identities one has to use the identity

$$\sum_{s=0}^{w}(2(\gamma-2\delta)s^3-3(\gamma-2\delta)s^2w+\gamma sw^2-\delta w^3)A(s)A(-s+w)=0, \tag{3.14}$$

where w is an arbitrary integer $w\geq 0$, δ and γ are arbitrary integers.

One can try to identify the generators of the Virasoro algebra *Vir*. The obvious strategy is to expand $T(z)$ in a power series around $z=0$. It results in

$$T(z) = \sum_{n=-\infty}^{+\infty} l_n z^{-n-2} \tag{3.15}$$

with

$$\begin{aligned} l_n = \tfrac{1}{2}((A(0))^{-1/2}L_{1,n} - \tfrac{1}{2}(A(0))^{-3/2}A(1)L_{1,n+1} + \cdots \\ + A(0)^{-1}L_{2,n} - (A(0))^{-2}A(1)L_{2,n+1} + \cdots). \end{aligned} \tag{3.16}$$

The argument leading to (3.16) is rather delicate. It is necessary to expand $1/y$ around $z = 0$ and it is unclear which branch of $y(z)$ one should take. Actually, in the hyperelliptic case both values differ only by sign. The generalization of (3.16) to the case of generic RS is rather problematic.

The perturbative computations confirm that the l_n satisfy the algebra

$$[l_n, l_m] = (n - m)l_{n+m} + \frac{c}{12}(n^3 - n) \tag{3.17}$$

with

$$c = \frac{\Gamma}{4} = \frac{\Gamma'}{4}. \tag{3.18}$$

The algebra (3.9)–(3.11) turns out to be symmetric under the transformations

$$L_{2,n} \to L_{2,-n-(2g+2)}, \tag{3.19}$$

$$L_{1,n} \to L_{1,-n-(g+1)}, \tag{3.20}$$

$$A(j) \to A(2g + 2 - j). \tag{3.21}$$

The term containing B in (3.10) should be subject to transformation which follows from the definition (3.12):

$$B(j) \to B(-2g - 2 - j). \tag{3.22}$$

The symmetry (3.19)–(3.21) can be verified by direct inspection. In the next section it will be explained why it is present.

4. Explicit realization

In the beginning of this section general RS are considered. The operator formalism of [18] gives rise to a natural notion of normal ordering. The operators are either creation and annihilation or they correspond to zero modes (there are M_s such operators $b_{s,j}$ for each value of s). Obvious normal ordering rules have to be supplemented requiring that zero mode excitations should be treated as creation operators.

The following oscillators are annihilation operators:

$$b_{k,j}|0\rangle = 0, \quad j \geq 1 - \lambda, \qquad c_{k,j'}|0\rangle = 0, \quad j' \geq \lambda \tag{4.1}$$

A vacuum state $|0\rangle$ can be constructed with the properties

$$\langle 0|0\rangle = 0, \tag{4.2}$$

$$\left\langle 0 \left| \prod_{s=0}^{N-1} \prod_{j=1-\lambda-M_s}^{-\lambda} b_{s,j} \right| 0 \right\rangle = 1. \tag{4.3}$$

The fact that the elementary excitations satisfy anticommutation rather than commutation relations implies that $|0\rangle$ should be of the form of "Dirac sea" or, in other words, a semi-infinite wedge product:

$$|0\rangle = \prod_{k=0}^{N-1} \beta_{k,1-\lambda} \wedge \beta_{k,2-\lambda} \wedge \beta_{k,3-\lambda} \wedge \cdots \tag{4.4}$$

It will be assumed that "sectors" of the theory labeled by different indices k commute. Elementary oscillators should act on vectors from the representation space according to

$$b_{k,j} \hookrightarrow \beta_{k,j} \wedge \cdots, \qquad c_{k,j} \hookrightarrow \frac{\partial}{\partial \beta_{k,-j}}. \tag{4.5}$$

A bilinear form reproducing (4.2) and (4.3) is given in the following way. For

$$|x\rangle = \prod_{k=0}^{N-1} \beta_{k,j_{k_1}} \wedge \beta_{k,j_{k_2}} \wedge \cdots, \tag{4.6}$$

$$|y\rangle = \prod_{k=0}^{N-1} \beta_{k,j'_{k_1}} \wedge \beta_{k,j'_{k_2}} \wedge \cdots, \tag{4.7}$$

one defines

$$\langle y|x\rangle \equiv \prod_{k=0}^{N-1} \cdots \wedge \beta_{k,-j'_{k_2}+H_k} \wedge \beta_{k,-j'_{k_1}+H_k} \wedge \beta_{k,j_{k_1}} \wedge \beta_{k,j_{k_2}} \wedge \cdots, \tag{4.8}$$

where $H_k = 1 - 2\lambda - M_k$. By definition, (4.8) is equal to zero unless all the excitations are present there. If not zero it is equal to 1.

This choice of the bilinear form implies

$$b^{\dagger}_{k,j} = b_{k,-j+H_k}, \qquad c^{\dagger}_{k,j} = c_{k,-j-H_k}. \tag{4.9}$$

This definition of conjugation may seem unnatural, but there is a clear explanation for that.

Let us consider from now on the hyperelliptic case. The numbers of b zero modes for $k = 0, 1$ are equal to

$$M_0 = \lambda(g+1) - 2\lambda + 1, \qquad M_1 = \lambda(g+1) - 2\lambda - g \tag{4.10}$$

so that

$$H_0 = -\lambda(g+1), \qquad H_1 = -(\lambda - 1)(g+1). \tag{4.11}$$

The meaning of normal ordering in (2.14) and (2.15) is now clear. From the definition of $L_{k,n}$ it is obvious that some of them are free from normal ordering ambiguities. This is true for all $L_{1,j}$ and also for $L_{2,j}$ with $j \geq 1$. Automatically, the same applies to l_j (generators of Virasoro algebra – see (3.16)) for $j \geq 1$.

The operators $L_{2,j}$ for $j \geq 1$ annihilate the vacuum state. One can choose the normal ordering constant in $L_{2,0}$ in such a way that it also annihilates vacuum

$$\begin{aligned} :L_{2,0}: = \sum_{k=0}^{1} \Bigg(& \sum_{m=1}^{2g+2} \sum_{j \leq m-\lambda} A(m) \left(\frac{(\lambda+k)m}{2} - j \right) b_{k,j} c_{k,m-j} \\ & - \sum_{m=1}^{2g+2} \sum_{j > m-\lambda} A(m) \left(\frac{(\lambda+k)m}{2} - j \right) c_{k,m-j} b_{k,j} \Bigg) \\ & + A(0) \sum_{k=0}^{1} \left(\sum_{j \geq 1-\lambda} j c_{k,-j} b_{k,j} - \sum_{j \leq -\lambda} j b_{k,j} c_{k,-j} \right). \end{aligned} \tag{4.12}$$

The normal ordering for the generators $L_{1,j}$ is not modified. The $L_{1,k}$'s for $k \geq 0$ annihilate vacuum state. In the standard operator formalism for CFTs on the complex sphere the definitions of bilinear form and conjugation are closely related with radial quantization. The complex sphere is equipped with a Euclidean metric and the time involution acts on the "space–time" coordinate z according to $z \to 1/\bar{z}$. There is a 1–1 correspondence between elements of the Hilbert space and the conformal fields

$$|\psi_{\text{in}}\rangle = \lim_{z,\bar{z} \to 0} \psi(z, \bar{z})|0\rangle. \tag{4.13}$$

A natural definition of conjugation is

$$\psi^{\dagger}(z, \bar{z}) = \bar{z}^{-2\lambda} z^{-2\bar{\lambda}} \psi \left(\frac{1}{\bar{z}}, \frac{1}{z} \right). \tag{4.14}$$

It is assumed that ψ is a conformal field of dimensions λ and $\bar{\lambda}$.

Applying the above rules also to the fields b–c one obtains, e.g.

$$\sum_{j=-\infty}^{+\infty} b_{0,j}^{\dagger} \frac{\bar{z}^{-\lambda-j}}{y^{\lambda}(\bar{z})} = \sum_{j=-\infty}^{+\infty} b_{0,j} \bar{z}^{-2\lambda} \frac{\bar{z}^{\lambda+j}}{y^{\lambda}(1/\bar{z})}. \tag{4.15}$$

It is very useful to define $\tilde{y}$ in the following manner:

$$y^2 \left(\frac{1}{z} \right) = \sum_{j=0}^{2g+2} A(j) z^{-j} = z^{-2g-2} \sum_{j=0}^{2g+2} \tilde{A}(j) z^{j} = z^{-2g-2} \tilde{y}^2(z) \tag{4.16}$$

with

$$\tilde{A}(j) = A(2g+2-j). \tag{4.17}$$

With that definition one can write

$$\sum_{j=-\infty}^{+\infty} b^{\dagger}_{0,j} \frac{\bar{z}^{-\lambda-j}}{y^{\lambda}(\bar{z})} = \sum_{j=-\infty}^{+\infty} b_{0,j} \frac{\bar{z}^{-\lambda+j+\lambda(g+1)}}{\tilde{y}^{\lambda}(\bar{z})}. \tag{4.18}$$

It is natural to define $b^{\dagger}_{0,j} = b_{0,-j-\lambda(g+1)}$. This conjugation has in general to be taken together with the transformation $A \to \tilde{A}$ (4.17). The definition of conjugation for $b_{k,j}$ and $c_{k,j}$ works perfectly well since the algebra (2.7) does not contain $A(j)$. On the other hand the algebra of $L_{k,j}$ includes explicitly $A(j)$ and so the conjugation for $L_{k,j}$

$$L^{\dagger}_{1,j} = L_{1,-j-(g+1)}, \qquad L^{\dagger}_{2,j} = L_{2,-j-(2g+2)}. \tag{4.19}$$

(Eq. (4.19) can be deduced in the same way as (4.18)) has to be supplemented by the transformation $A(j) \to \tilde{A}(j)$.

5. Possible generalizations

The motivation to develop the monodromy formalism is a hope to apply it to a generic RS given by (1.1). Let us describe in general terms what the features of such construction should be.

One can use the operator formalism for b–c system and the elementary excitations to define $L_{k,j}$ as it was done in Section 2. There are several choices for the monodromy decomposition of $T(z)$. One possibility is

$$T(z) = \sum_{k=0}^{N-1} \sum_{j=-\infty}^{+\infty} L_{k,j} \frac{z^{-j-2}}{y^k}\, \mathrm{d}z^2. \tag{5.1}$$

The other possibility is suggested by the form of expansions for b field (1.4)

$$T(z) = \sum_{k=0}^{N-1} \sum_{j=-\infty}^{+\infty} L_{k,j} \frac{z^{-j-2} y^{2-k}}{(F_y)^2}\, \mathrm{d}z^2. \tag{5.2}$$

Whatever the monodromy decomposition is, one gets concrete expressions for $L_{k,j}$ and the whole algebra can be calculated. Its general features can be read from OPE. Let us concentrate on the case (5.1). The requirement that the OPE keeps the monodromy properties of the components of $T(z)$ implies a general structure which can be written in symbolic notation. Let us present more detailed formulas in the case of a genus 3 RS described by the algebraic equation

$$y^3 + 3yP(z) - 2Q(z) = 0. \tag{5.3}$$

$P(z)$ is a polynomial of degree 3 and $Q(z)$ a polynomial of degree 4. Monodromy implies

$$\begin{aligned} &T_0T_0 \sim T_0 + \text{c.e. term}, \quad T_0T_1 \sim T_1, \quad T_0T_2 \sim T_2, \quad T_1T_1 \sim T_2, \\ &T_1T_2 \sim T_0 + T_2 + \text{c.e. term}, \qquad T_2T_2 \sim T_0 + T_1 + T_2 + \text{c.e. term}. \end{aligned} \tag{5.4}$$

The meaning of the above symbolic notation is that for example

$$T_0(z)T_1(w) \sim \frac{\alpha T_1(w)}{(z-w)^2} + \frac{\beta \partial T_1(w)}{z-w}, \tag{5.5}$$

where α and β are constants.

The last two relations in (5.4) follow from the identities:

$$\frac{1}{y^3} = \frac{1}{2Q(z)} + \frac{1}{y^2}\frac{3P(z)}{2Q(z)}, \tag{5.6}$$

$$\frac{1}{y^4} = \frac{3P(z)}{4Q^2(z)} + \frac{1}{y}\frac{1}{2Q(z)} + \frac{1}{y^2}\frac{9P^2(z)}{4Q^2(z)}. \tag{5.7}$$

The generators $L_{k,j}$ are expressible in terms of $T_k(z)$ by means of loop integrals (before fixing the constants α, β etc. it is better to call them $\tilde{L}_{k,j}$)

$$\tilde{L}_{k,j} = \frac{1}{2\pi \mathrm{i}} \oint T_k y^k z^{j+1}. \tag{5.8}$$

The integral (5.8) is well defined as the monodromy property of the product of $T_k y^k$ (no summation) are trivial. From (5.4) the commutation relations among the $L_{k,j}$'s can be computed.

The conjugation $L^\dagger_{k,j}$ can be calculated either directly in terms of $b^\dagger_{k,j}$ and $c^\dagger_{k,j}$ or by applying directly to (5.1) the radial quantization techniques. It is clear that conjugation has to act also on the parameters of the algebraic equation. The simple transformation $A(j) \to A(2g+2-j)$ obtained for hyperelliptic curve will be replaced by a more complicated one.

One should analyze in detail, e.g. genus 3 RS. Emphasis has to be put on the choice of the simplest possible monodromy decomposition of $T(z)$, perhaps one analogous to the Z_3 symmetric case.

$$T(z) = \sum_{k=0}^{2} \sum_{j=-\infty}^{+\infty} L_{k,j} z^{-j-2} (S(z, y(z)))^k \, \mathrm{d}z^2, \tag{5.9}$$

where $S(z, y(z))$, which has nontrivial monodromy, satisfies the equation $S^3 = W(z)$, $W(z)$ being rational function of z (i.e. S^3 has trivial monodromy).

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